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Supersymmetric quantum mechanics and the inverse scattering method

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Abstract. The procedures for finding a new potential (1) by eliminating the ground state of a given potential (2) by adding a bound state below the ground state of a given potential and (3) by generating the phase equivalent family of a given potential using the supersymmetric pairing of the spectra of the operators A^+A^- and A^-A^+ are compared with the application of the Gelfand-Levitan procedure for the corresponding cases. It is shown how the equivalence of the two procedures may be established. A distinction is made between the modifications of the Jost functions associated with four different types of transformations generated by the concept of a supersymmetric partner to a given Schrödinger equation. It is shown that the Bargmann class of potentials may be generated using suitable combinations of the four types of transformations.

1. Introduction

In the preceding paper (Sukumar 1985b, hereafter referred to as I) it was shown that by using the idea of a supersymmetric partner to a Hamiltonian function H_1 of a single variable x, it is possible to find another Hamiltonian H_2 which has one of the following features: either (i) the complete spectrum of H_2 is made up of all eigenvalues of H_1 except the ground state of H_1, or (ii) the complete spectrum of H_2 is made up of all eigenvalues of H_1 and in addition one further eigenvalue which lies below the ground state of H_1, or (iii) the spectrum of H_2 is identical to that of H_1. It was shown in I that in all three cases the eigenfunctions of H_1 and H_2 for the common eigenvalues are connected by a linear differential operator. By repeated application of this procedure of either deleting an eigenvalue or adding an eigenvalue or maintaining the same eigenvalues it is possible to generate Hamiltonians whose spectra bear definite relationships to each other. The inverse scattering theory can also accomplish the same tasks through solving either the Gelfand-Levitan or the Marchenko integral equations (Abraham and Moses 1980, Nieto 1984, Mielnick 1984). The aim of this paper is to elucidate the relationship between the two approaches. It will also be shown that the families of potentials generated by the application of supersymmetric quantum mechanics are members of the Bargmann class of potentials (Bargmann 1949).

The radial Schrödinger equation differs from the Schrödinger equation in the space $[-\infty, \infty]$ in essential respects. The boundary conditions on the eigenfunctions and the allowed singularities of the potential V are different for the two spaces $[-\infty, \infty]$ and $[0, \infty]$. In § 2 of this paper the modifications from I introduced by switching from x to r are considered and the Jost function modifications corresponding to four different types of transformations are studied. Section 3 contains a discussion of the relationship of the four types of transformations to the Bargmann class of potentials. In § 4 several recent applications of the Gelfand-Levitan procedure for solving the inverse scattering problem are considered and it is shown that each of these applications is equivalent to an appropriate combination of the four types of transformations generated by the algebra of supersymmetry. Section 5 contains the conclusions.

2. Modifications from Sukumar (1985b)

2.1. *The radial Schrödinger equation*

In this paper we consider the radial Schrödinger equation with the Hamiltonian

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)$$
$$V(r) = \frac{l(l+1)}{2r^2} + U(r). \tag{1}$$

The potential $V(r)$ is assumed to be regular, not singular. Specifically, the potentials discussed in this paper are restricted to be no more singular than $1/r^2$ at the origin and decreasing at least as fast as $1/r^2$ as $r \to \infty$.

In this paper the term 'normalisation constant of the eigenfunction' will be used often. This term has a specific meaning in the terminology of the inverse scattering theory. All bound-state eigenfunctions are understood to be normalised to unity in the usual way to reflect the condition that the total probability of finding the bound particle somewhere in space should be unity. However, in the inverse scattering method the term 'normalisation' is used in a different sense. The regular solution φ of the radial Schrödinger equation is defined to be a solution that satisfies the boundary condition

$$\lim_{r\to 0} \varphi(r, E, l) = \frac{r^{l+1}}{(2l+1)!!}. \tag{2}$$

The regular solution will grow exponentially as $r \to \infty$ when E is not one of the eigenenergies. However, when E is one of the eigenenergies $E^{(i)}$ the bound-state eigenfunction, which decreases exponentially as $r \to \infty$, is proportional to the regular solution

$$\psi(r, E^{(i)}, l) = \alpha\varphi(r, E^{(i)}, l) \tag{3a}$$

where

$$\int_0^\infty \psi^2 \, \mathrm{d}r = 1. \tag{3b}$$

It is this proportionality constant α that corresponds to the 'normalisation constant' referred to in the inverse scattering method. Throughout this paper the term 'normalisation' will be used in the sense in which it is used in the inverse scattering theory. The term 'normalisable' will, however, be used in the usual sense of the word, i.e. $\int_0^\infty \psi^2 \, \mathrm{d}r$ is finite.

In view of the different types of transformations of the radial equation that will be discussed in this paper, the following notations will be adopted. The eigenfunctions of H defined in equation (1) are denoted by $\psi^{(i)}$ for the discrete states at energy $E^{(i)}$, the phaseshifts for the continuum states $\psi(r, E)$ for positive energies $E=\frac{1}{2}k^2$ are denoted by $\delta(l, k)$ and the Jost function by $F(l, k)$. The potentials, eigenstates, phaseshifts and Jost function after the transformation are denoted by adding a tilde, $\tilde{\psi}(r, E)$, for example. The different types of transformations are distinguished by adding a suffix, $\tilde{\psi}_1(r, E)$, for example. Successive transformations are indicated by adding further suffixes and tildes.

2.2. *Jost function*

The integral representation of the Jost function for a potential $U(r)$ with N bound states at energies $E=E^{(i)}$ and scattering phaseshifts $\delta(l, k)$ at energies $E=\frac{1}{2}k^2$ for angular momentum l is given by (see Chadan and Sabatier 1977, for example)

$$F(l, k)=\prod_{i=1}^{N}\left(1-\frac{E^{(i)}}{E}\right)\exp\left(-\frac{2}{\pi}\int_0^\infty \frac{\delta(l, p)p\,\mathrm{d}p}{p^2-k^2}\right). \tag{4}$$

The phase of the Jost function is $-\delta(l, k)$ while the modulus is given by

$$|F(l, k)|=\prod_{i=1}^{N}\left(1-\frac{E^{(i)}}{E}\right)\exp\left(-\frac{2}{\pi}P\int_0^\infty \frac{\delta(l, p)p\,\mathrm{d}p}{p^2-k^2}\right) \tag{5}$$

where the symbol P stands for principal value. The spectral density for positive energies is given by

$$\frac{\mathrm{d}P(E)}{\mathrm{d}E}=\frac{E^{l+1/2}}{\pi}|F(l, k)|^{-2}. \tag{6}$$

Knowledge of the phaseshifts for all positive energies, the bound-state energies $E^{(i)}$ and the normalisation constants $C^{(i)}$ associated with each of the bound states enables the complete determination of the potential $U(r)$.

2.3. *Elimination of the ground state of V*

By the methods of I it can be shown that H defined by equation (1) has a supersymmetric partner

$$\tilde{H}_1=H-(\mathrm{d}^2/\mathrm{d}r^2)\ln\psi^{(0)}(r). \tag{7}$$

Since

$$\lim_{r\to 0}\psi^{(0)}(r)\sim r^{l+1} \tag{8}$$

H corresponds to the potential

$$\tilde{V}_1(r)=\frac{(l+1)(l+2)}{2r^2}+U(r)-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\psi^{(0)}(r)}{r^{l+1}}\right) \tag{9}$$

where the singularity at the origin has been separated to show that $\tilde{V}_1(r)$ corresponds to angular momentum $(l+1)$. The spectral mapping can be shown using the analysis of I to be

$$\tilde{E}_1^{(m)}=E^{(m+1)}, \qquad m=0, 1, 2, \ldots \tag{10a}$$

$$\tilde{\psi}_1^{(m)}=(E^{(m+1)}-E^{(m)})^{-1/2}A_1^-\psi^{(m+1)} \tag{10b}$$

where

$$A_1^- = \frac{1}{\sqrt{2}}\left[-\frac{d}{dr} + \left(\frac{d}{dr}\ln \psi^{(0)}(r)\right)\right]. \tag{10c}$$

Extension of the above eigenfunction relation to positive energy states and use of the asymptotic forms

$$\lim_{r\to\infty} \psi(r, E) \sim \sin[kr - \tfrac{1}{2}l\pi + \delta(l, k)] \tag{11a}$$

and

$$\lim_{r\to\infty} \psi^{(0)}(r) \sim \exp(-\gamma^{(0)} r) \tag{11b}$$

then gives

$$\lim_{r\to\infty} \tilde{\psi}_1(r, E) \sim \sin[kr - \tfrac{1}{2}(l+1)\pi + \tilde{\delta}_1(l+1, k)] \tag{12a}$$

where

$$\tilde{\delta}_1(l+1, k) = \delta(l, k) - \tan^{-1}(\gamma^{(0)}/k) \tag{12b}$$

$$E = \tfrac{1}{2}k^2 \qquad \text{and} \qquad E^{(0)} = (\tfrac{1}{2}\gamma^{(0)^2}). \tag{12c}$$

We illustrate this phaseshift relation with an example. When $U(r) = -b/r$, i.e. for the Coulomb potential, it was shown (Sukumar 1985a) that the potential $\tilde{U}(r)$ obtained by eliminating the ground state of $V(r)$ is also a Coulomb potential. Equation (12b) then gives a relation between the Coulomb phaseshifts δ_c for the angular momenta l and $(l+1)$. It is easy to show that equation (12b) leads to

$$\delta_c(l+1, k) = \delta_c(l, k) - \tan^{-1}[\eta/(l+1)] \tag{13a}$$

where

$$\eta = b/k. \tag{13b}$$

This phaseshift relation is satisfied by the well known expression for the Coulomb phase shifts which is given by

$$\exp[2i\delta_c(l, k)] = \frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)}. \tag{14}$$

Equations (10a), (4) and (5) enable the establishment of the following relationship between the Jost functions for the potentials $\tilde{V}_1(r, l+1)$ and $V(r, l)$:

$$\frac{F(l, k)}{\tilde{F}_1(l+1, k)} = \left(1 - \frac{E^{(0)}}{E}\right) \exp[-i(\delta(l, k) - \tilde{\delta}_1(l+1, k)]$$
$$\times \exp\left(-\frac{2}{\pi} P \int_0^\infty \frac{\delta(l, p) - \tilde{\delta}_1(l, p)}{p^2 - k^2} p\, dp\right). \tag{15}$$

The principal value integral can be evaluated using the integral relation (Gradshteyn and Ryzhik 1965)

$$\frac{4}{\pi} P \int_0^\infty \frac{\cot^{-1}(p/\gamma)}{p^2 - k^2} p\, dp = \ln(1 + \gamma^2/k^2). \tag{16}$$

Therefore, equations (12b)-(15) give

$$\frac{\tilde{F}_1(l+1, k)}{F(l, k)} = \frac{k}{(k - i\gamma^{(0)})}. \tag{17}$$

Thus the elimination of the ground state of V by the supersymmetric method is equivalent to multiplying the Jost function by the factor $[k/(k - i\gamma^{(0)})]$ and changing the angular momentum from l to $(l+1)$.

2.4. *Addition of bound state*

The potential $\tilde{V}_2$ with a ground state at $\tilde{E} = -\frac{1}{2}\tilde{\gamma}^2 < E^{(0)}$, i.e. below the ground state of V, in addition to sharing all the eigenvalues of V can be constructed by the methods of I. Since the potential in the radial equation can have singularities of the form $1/r^2$ the equations in I must be recast in an appropriate form. The regular solution in the potential V at energy E denoted by φ satisfies

$$\lim_{r \to 0} \varphi \sim r^{l+1} \qquad \text{and} \qquad \lim_{r \to \infty} \varphi \sim \exp(\tilde{\gamma} r). \tag{18}$$

Since the energy E is below the ground state of V, φ is nodeless for $r > 0$ and may be chosen to be positive definite for $r > 0$ (see the appendix). The linearly independent solution can be taken to be

$$f(r) = \varphi(r) \int_r^{\infty} dz/\varphi^2(z). \tag{19}$$

It is easy to show that

$$\lim_{r \to 0} f(r) \sim r^{-l} \qquad \text{and} \qquad \lim_{r \to \infty} f(r) \sim \exp(-\tilde{\gamma} r). \tag{20}$$

f is one of the Jost solutions (see Newton 1966, for example) defined by a boundary condition in the asymptotic region. For energies $\tilde{E} < E^{(0)}$, f is also a nodeless function and is positive definite (see the appendix). When E is not only less than $E^{(0)}$ but also less than the absolute minimum of the potential V, the positivity of $(V - \tilde{E})$ guarantees that φ and f are monotonically growing functions of r in the directions $r = 0 - \infty$ and $r = \infty - 0$, respectively. When $V_{\min} < \tilde{E} < E^{(0)}$, φ and f are no longer monotonically growing functions but nevertheless remain nodeless. These assertions on the behaviour of φ and f are shown to be true by explicit construction of φ and f in the appendix. The function

$$\psi = \varphi \cos\theta + f \sin\theta \tag{21}$$

is also a nodeless function when $0 < \theta < \pi/2$ and $1/\psi$ is a normalisable function for this range of values of θ since

$$\lim_{r \to 0} 1/\psi = \lim_{r \to 0} 1/(f \sin\theta) \sim r^l \tag{22a}$$

and

$$\lim_{r \to \infty} 1/\psi = \lim_{r \to \infty} 1/(\varphi \cos\theta) \sim \exp(-\tilde{\gamma} r). \tag{22b}$$

By the methods of I, it is easy to infer that when $0 < \theta < \pi/2$,

$$\tilde{V}_2(r) = V(r) - \frac{d^2}{dr^2} \ln \psi(r, \tilde{E}, \theta) \tag{23a}$$

with

$$\psi(r, \tilde{E}, \theta) = \varphi(r, \tilde{E})\left(\cos\theta + \sin\theta \int_r^\infty \mathrm{d}z/\varphi^2(z, \tilde{E})\right) \tag{23b}$$

has a ground state at

$$\tilde{E}_2^{(0)} = \tilde{E} \tag{24a}$$

with eigenfunction

$$\tilde{\psi}_2^{(0)}(r, \tilde{E}, \theta) = 1/\psi(r, \tilde{E}, \theta). \tag{24b}$$

The excited states are given by

$$\tilde{E}_2^{(m+1)} = E^{(m)}, \qquad m = 0, 1, 2, \ldots \tag{25a}$$

$$\tilde{\psi}_2^{(m+1)} = -(E^{(m)} - \tilde{E})^{-1/2} A_2^- \psi^{(m)} \tag{25b}$$

where

$$A_2^-(\tilde{E}, \theta) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\psi(r, \tilde{E}, \theta)\right)\right]. \tag{25c}$$

If the potential V corresponds to angular momentum l as in equation (1), equation ($22a$) then gives

$$\tilde{V}_2(r, \tilde{E}, \theta) = \frac{l(l-1)}{2r^2} + U(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\psi(r, \tilde{E}, \theta) \tag{26}$$

which shows that $\tilde{V}_2$ corresponds to angular momentum $(l-1)$. It is clear that l must satisfy $l \geq 1$ for $\tilde{V}_2$ to be free of attractive $1/r^2$ singularities. Extension of equation ($25b$) to the positive energy states and use of the asymptotic forms

$$\lim_{r\to\infty} \psi(r, E) \sim \sin[kr - \tfrac{1}{2}l\pi + \delta(l, k)] \tag{11a}$$

$$\lim_{r\to\infty} \psi(r, \tilde{E}, \theta) \sim \exp(\tilde{\gamma} r) \tag{27}$$

then gives

$$\lim_{r\to\infty} \tilde{\psi}_2(r, E, \theta) \sim \sin[kr - \tfrac{1}{2}(l-1)\pi + \tilde{\delta}_2(l-1, k)] \tag{28a}$$

where

$$\tilde{\delta}_2(l-1, k) = \delta(l, k) + \tan^{-1}(\tilde{\gamma}/k). \tag{28b}$$

Equation ($28b$) shows that all members of the family $\tilde{V}_2(r, \tilde{E}, \theta)$ lead to identical phaseshifts for a fixed energy E for $0 < \theta < \frac{1}{2}\pi$. Furthermore, since

$$\lim_{r\to 0} \frac{\mathrm{d}}{\mathrm{d}r}\ln\psi(r, \tilde{E}, \theta) = -l/r \tag{29}$$

equation ($25b$) shows that for a fixed principal quantum number m, $\lim_{r\to 0} \tilde{\psi}_2^{(m+1)}(r, \tilde{E}, \theta)$ is independent of θ and therefore the excited states of $\tilde{V}_2(r, \tilde{E}, \theta)$

for various θ have identical normalisations. However, the normalised ground-state eigenfunction

$$\tilde{\psi}_2^{(0)}(r, \tilde{E}, \theta) = \frac{(\sin\theta\cos\theta)^{1/2}}{\varphi(\cos\theta + \sin\theta \int_r^\infty \mathrm{d}z/\varphi^2)} \tag{30a}$$

$$\int_0^\infty (\tilde{\psi}_2^{(0)})^2 \,\mathrm{d}r = 1, \tag{30b}$$

shows that

$$\lim_{r\to 0} \tilde{\psi}_2^{(0)}(r, \tilde{E}, \theta) = \left(\frac{\cos\theta}{\sin\theta}\right)^{1/2} r^l. \tag{31}$$

Hence the ground-state eigenfunctions of the family of potentials $\tilde{V}_2(r, \tilde{E}, \theta)$ have different normalisations, i.e. different proportionalities to the regular solution, although they belong to the same eigenvalue. It has been shown that the phaseshifts, the eigenvalues and the normalisation constants of the excited states are identical for all members of the family of potentials $\tilde{V}_2(r, \tilde{E}, \theta)$, $0 < \theta < \frac{1}{2}\pi$, while the ground-state eigenfunctions belonging to the eigenvalue $\tilde{E}_2^{(0)} = \tilde{E}$ have different normalisation constants for different values of θ. Clearly the family $\tilde{V}_2(r, \tilde{E}, \theta)$ is an example of the phase equivalent family first discussed by Bargmann.

The phaseshifts and the bound-state energies of V and $\tilde{V}_2$ enable the comparison to the corresponding Jost functions. From equations (4), (28*b*) and (16) it is easy to show that

$$\frac{\tilde{F}_2(l-1, k)}{F(l, k)} = \frac{k - \mathrm{i}\tilde{\gamma}}{k}. \tag{32}$$

2.5. *Boundary values of θ and equivalent potentials*

When the parameter θ lies outside the range $0 < \theta < \frac{1}{2}\pi$, the wavefunction ψ in equation (21) does not lead to a normalisable $1/\psi$. When $-\pi < \theta < 0$ or $\pi > \theta > \frac{1}{2}\pi$, ψ vanishes at some finite value of r because either $\sin\theta$ or $\cos\theta$ assumes negative values and $\int_r^\infty \mathrm{d}z/\varphi^2$ can take all values from 0 to ∞. If ψ vanishes at a finite value of r then $(\mathrm{d}^2/\mathrm{d}r^2)\ln\psi$ diverges at this point. This then would lead to a singular V. However, the critical values $\theta = 0$ and $\theta = \frac{1}{2}\pi$ must be studied separately.

(*a*) When $\theta = 0$

$$\psi(r, \tilde{E}, 0) = \varphi \qquad \lim_{r\to 0}\psi \sim r^{l+1} \qquad \lim_{r\to\infty}\psi \sim \exp(\tilde{\gamma} r). \tag{33}$$

The vanishing value of ψ at $r = 0$ shows that $1/\psi$ is not normalisable, but

$$\lim_{r\to 0}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\varphi \sim -\frac{(l+1)}{r^2} \qquad \lim_{r\to\infty}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\varphi \sim 0. \tag{34}$$

The positivity of φ guarantees that there are no singularities in $(\mathrm{d}^2/\mathrm{d}r^2)\ln\varphi$ for $r > 0$. These conditions ensure that it is possible to find a non-singular supersymmetric partner to V. It is easy to show that

$$\tilde{V}_3(r, \tilde{E}) = \frac{(l+1)(l+2)}{2r^2} + U(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\varphi(r, \tilde{E})}{r^{l+1}}\right). \tag{35}$$

$\tilde{V}_3$ corresponds to angular momentum $(l+1)$ and has a spectrum identical to that of V:

$$\tilde{E}_3^{(m)} = E^{(m)} \qquad m = 0, 1, 2, \ldots . \tag{36a}$$

The eigenfunction relation may be chosen to be

$$\tilde{\psi}_3^{(m)} = (E^{(m)} - \tilde{E})^{-1/2} A_3^- \psi^{(m)} \tag{36b}$$

with

$$A_3^-(\tilde{E}) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln \varphi(r, \tilde{E})\right)\right]. \tag{36c}$$

The phaseshifts in the potentials V and $\tilde{V}_3$ are related by

$$\tilde{\delta}_3(l+1, k) = \delta(l, k) + \tan^{-1}(\tilde{\gamma}/k). \tag{37}$$

The family of potentials $\tilde{V}_3(r, \tilde{E})$ for different values of $\tilde{E} < E^{(0)}$ have identical spectra but different phaseshifts for the same energy. Furthermore $\lim_{r\to 0} \tilde{\psi}_3^{(m)}(r, \tilde{E})$ is different for different values of $\tilde{E}$, i.e. for a fixed m the normalisation constants of the states $\tilde{\psi}_3^{(m)}(r, \tilde{E})$ depend on $\tilde{E}$ because of the factor $(E - \tilde{E})$ in equation (36b). Therefore, the family of potentials $\tilde{V}_3(r, \tilde{E})$ for different E do not belong to the phase equivalent class. The Jost functions for the potentials V and $\tilde{V}_3$ can be shown to be related in the manner

$$\frac{\tilde{F}_3(l+1, k)}{F(l, k)} = \frac{k}{k + \mathrm{i}\tilde{\gamma}}. \tag{38}$$

(b) When $\theta = \frac{1}{2}\pi$

$$\psi(r, \tilde{E}, \tfrac{1}{2}\pi) = f \qquad \lim_{r\to 0} \psi \sim r^{-l} \qquad \lim_{r\to\infty} \psi \sim \exp(-\tilde{\gamma} r). \tag{39}$$

Hence $\theta = \frac{1}{2}\pi$ does not lead to normalisable $1/\psi$ because $1/\psi$ diverges as $r \to \infty$. However,

$$\lim_{r\to 0} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln f = \frac{l}{r^2} \qquad \lim_{r\to\infty} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln f \sim 0. \tag{40}$$

These conditions, together with the absence of any other singularities of $(\mathrm{d}^2/\mathrm{d}r^2) \ln f$, ensure that a singularity-free supersymmetric partner to V may be constructed. Thus

$$\tilde{V}_4(r, \tilde{E}) = \frac{l(l-1)}{2r^2} + U(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln[r^l f(r, \tilde{E})] \tag{41}$$

corresponds to angular momentum $(l-1)$ and has a spectrum identical to that of V:

$$\tilde{E}_4^{(m)} = E^{(m)} \qquad m = 0, 1, 2, \ldots . \tag{42a}$$

The eigenfunctions are linked by

$$\tilde{\psi}_4^{(m)} = -(E^{(m)} - \tilde{E})^{-1/2} A_4^- \psi^{(m)} \tag{42b}$$

with

$$A_4^-(\tilde{E}) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln f(r, \tilde{E})\right)\right]. \tag{42c}$$

The phaseshifts can be shown to be related by

$$\tilde{\delta}_4(l-1, k) = \delta(l, k) - \tan^{-1}(\tilde{\gamma}/k). \tag{43}$$

The family of potentials $\tilde{V}_4(r, \tilde{E})$ for different values of $\tilde{E}$ have identical spectra, but different phaseshifts for the same energy E. $\tilde{V}_4(r, \tilde{E})$ therefore do not belong to the phase equivalent family. The Jost functions for the potentials V and $\tilde{V}_4$ can be shown to be related as

$$\frac{\tilde{F}_4(l-1, k)}{F(l, k)} = \frac{(k + \mathrm{i}\tilde{\gamma})}{k}. \tag{44}$$

The above discussion for the limiting values of $\theta = 0$ and $\theta = \frac{1}{2}\pi$ shows that the potentials $V(r)$, $\tilde{V}_3(r, \tilde{E})$ and $\tilde{V}_4(r, \tilde{E})$ defined by equations (1), (35) and (41), respectively, have identical spectra. With the restriction that $\tilde{E}$ should be less than the ground-state eigenvalue of V, the two families of potentials $\tilde{V}_3(r, \tilde{E})$ and $\tilde{V}_4(r, \tilde{E})$ with $-\infty < \tilde{E} < E^{(0)}$ have been shown to have spectra identical to that of V but with different phaseshifts for the different members of each family $\tilde{V}_3$ and $\tilde{V}_4$.

3. Relation of the four types of transformations to the Bargmann potentials

3.1. Summary

In § 2 it was shown that by a suitable factorisation of the radial Schrödinger equation, it is possible to discover an underlying supersymmetric algebra. This algebra (Witten 1981) may be used to generate four different types of transformations of the radial Schrödinger equation. The four transformations may be classified as follows.

(1) T_1 is a transformation that eliminates the ground state $[E^{(0)}, \psi^{(0)}]$ of the potential $V(r)$, changes the angular momentum from l to $(l+1)$ and leaves the rest of the eigenvalue spectrum of V unaltered. T_1 also changes the Jost function corresponding to V by the multiplicative factor $k/(k - \mathrm{i}\gamma^{(0)})$ where $\gamma^{(0)} = [-2E^{(0)}]^{1/2}$. The new eigenfunctions in the potential (9) are given by equations (10).

(2) T_2 is a transformation that adds a bound state $[\tilde{E}_2^{(0)}, \tilde{\psi}_2^{(0)}]$ below the ground state $E^{(0)} > \tilde{E}_2^{(0)}$ of V, changes the angular momentum from l to $(l-1)$ but leaves the rest of the eigenvalue spectrum of V unaltered. T_2 also changes the Jost function for V by the multiplicative factor $(k - \mathrm{i}\tilde{\gamma}_2^{(0)})/k$ where $\tilde{\gamma}_2^{(0)} = [-2\tilde{E}_2^{(0)}]^{1/2}$. The new eigenfunctions in the potential (26) are given by equations (24) and (25).

(3) T_3 is a transformation that maintains the eigenvalue spectrum of V unaltered but changes the angular momentum from l to $(l+1)$. T_3 also changes the Jost function for V by the multiplicative factor $k/(k + \mathrm{i}\tilde{\gamma})$ where $\tilde{\gamma} = [-2\tilde{E}]^{1/2}$ and $\tilde{E} < E^{(0)}$. The new eigenfunctions in the potential (35) are given by equations (36).

(4) T_4 is a transformation that leaves the eigenvalue spectrum of V unaltered but changes the angular momentum from l to $(l-1)$. T_4 also changes the Jost function for V by the multiplicative factor $(k + \mathrm{i}\tilde{\gamma})/k$ where $\tilde{\gamma} = (-2\tilde{E})^{1/2}$ and $\tilde{E} < E^{(0)}$. The new eigenfunctions in the potential (41) are given by equations (42).

In the one-dimensional case $-\infty < x < \infty$, singularities in the potential at $x = 0$ are not permitted and the physical wavefunctions are defined by boundary conditions at $x = \pm\infty$. In the case of the radial equation the boundary conditions on the eigenfunctions at $r = 0$ and at $r = \infty$ are different. All four types of transformations listed above have analogues in the space $[-\infty, \infty]$ but with the difference that no singularities of the type $1/x^2$ should be introduced by the transformations because of the boundary conditions usually imposed on $\psi(x)$. The transformations T_3 and T_4 may be distinguished as follows.

Let $\xi(x, \tilde{E})$ and $\xi'(x, \tilde{E})$ be solutions of the Schrödinger equation for $V(x)$ at energy $\tilde{E} < E^{(0)}$ with the boundary conditions

$$\lim_{x \to -\infty} \xi(x, \tilde{E}) = 0 \qquad \lim_{x \to +\infty} \xi'(x, \tilde{E}) = 0. \tag{45}$$

For $\tilde{E} < E^{(0)}$, the solutions ξ and ξ' are nodeless and satisfy

$$\lim_{x \to +\infty} \xi(x, \tilde{E}) \sim \exp(\tilde{\gamma} x) \qquad \lim_{x \to -\infty} \xi'(x, \tilde{E}) \sim \exp(\tilde{\gamma}|x|) \tag{46a}$$

where

$$\tilde{\gamma} = (-2\tilde{E})^{1/2}. \tag{46b}$$

Therefore T_3 generates

$$\tilde{V}_3(x, \tilde{E}) = V(x) - \frac{d^2}{dx^2} \ln \xi(x, \tilde{E}) \tag{47}$$

and T_4 generates

$$\tilde{V}_4(x, \tilde{E}) = V(x) - \frac{d^2}{dx^2} \ln \xi'(x, \tilde{E}). \tag{48}$$

$\tilde{V}_3$ and $\tilde{V}_4$ correspond to the limiting values $\alpha = +\infty$ and $\alpha = -\beta$ in the discussion in I. The correspondence with I may be established by noting that

$$\alpha = +\infty \to \xi(x, \tilde{E}) = \varphi_1 \int_{-\infty}^{x} dz/\varphi_1^2 \tag{49a}$$

$$\alpha = -\beta \to \xi'(x, \tilde{E}) = \varphi_1 \int_{\infty}^{x} dz/\varphi_1^2. \tag{49b}$$

In I $\varphi_1(x, \tilde{E})$ was assumed to be a nodeless solution in the potential $V(x)$ that diverges at both $x = +\infty$ and $x = -\infty$. When the potential V is a symmetric function of x, the two potentials $\tilde{V}_3$ and $\tilde{V}_4$ are connected by the parity transformation. It is easy to show that when V is an even function of x,

$$\tilde{V}_3(x, \tilde{E}) = \tilde{V}_4(-x, \tilde{E}). \tag{50}$$

The above analysis clarifies the relationship between the transformations in the spaces $[-\infty, \infty]$ and $[0, \infty]$.

3.2. *Relationship to Bargmann potentials*

It has been shown above that each of the transformations T_1–T_4 corresponds to a multiplication of the Jost function by a specific rational function of k. By repeated application of a combination of the four types of transformations in an appropriate order, the Jost function of V can be modified by any rational function of k. The generation of the Bargmann potentials (Bargmann 1949) corresponds to such a modification of the Jost function. Therefore it is clear that the Bargmann class of potentials may be generated by a suitable combination of T_1, T_2, T_3 and T_4. For example, the multiplication of the Jost function by a factor $(k + ib)/(k + ia)$ can be broken down into the two steps, multiplication by $(k + ib)/k$ followed by a further multiplication by $k/(k+ia)$, corresponding to application of T_4 followed by T_3. The physically acceptable Jost functions must satisfy the condition

$$\lim_{k\to\infty} F(l,k)=1 \tag{51a}$$

and the symmetry relation

$$F(l,-k^*)=[F(l,k)]^*. \tag{51b}$$

The modifications of F introduced by T_1-T_4 clearly satisfy these conditions.

The above analysis demonstrates that all physically acceptable modifications of the Jost function which are of the form of a multiplication by a rational function of k may be accomplished by a suitable combination of the four types of transformations T_1-T_4. In the next section the relationship of these transformations to the Gelfand-Levitan method (Gelfand and Levitan 1951) is studied.

4. Some recent applications of the Gelfand–Levitan procedure

4.1. Elimination of the ground state by the Gelfand-Levitan procedure

Abraham and Moses (1980) have applied the Gelfand-Levitan method to eliminate the ground state of V at energy $E^{(0)}$ without altering all the other bound-state energies, the normalisation constants and the spectral density for positive energies. Equation (6) shows that the spectral density for positive energies is governed only by the modulus of the Jost function. Hence elimination of a bound state without altering the spectral density for positive energies corresponds to the modification of the Jost function in the manner

$$\frac{F_{\text{new}}(l,k)}{F_{\text{old}}(l,k)}=\frac{(k+i\gamma^{(0)})}{(k-i\gamma^{(0)})} \tag{52}$$

(see Chadan and Sabatier (1977, p 55) for example). The angular momentum is also assumed to be unaltered by the transformations discussed by Abraham and Moses. The analysis in § 3.1 shows that the transformation implied by equation (52) may be achieved in two steps.

Step 1. Eliminate the ground state of V by a transformation of type T_1. Starting from the potential defined in equation (1), T_1 then generates

$$\tilde{V}_1(r)=\frac{(l+1)(l+2)}{2r^2}+U(r)-\frac{d^2}{dr^2}\ln\left(\frac{\psi^{(0)}(r)}{r^{l+1}}\right). \tag{9}$$

After application of T_1 the modified eigenstates and phase shifts are given by equations (10) and (12). The modification of the Jost function is given by

$$\frac{\tilde{F}_1(l+1,k)}{F(l,k)}=\frac{k}{(k-i\gamma^{(0)})}. \tag{17}$$

$\tilde{V}_1$ corresponds to angular momentum $(l+1)$.

Step 2. To reduce the angular momentum from $(l+1)$ to l without altering the spectrum, a suitable transformation of type T_4 can be found by choosing the energy E in the transformation equations for T_4 to be equal to $E^{(0)}$, the ground-state energy of V. $T_4(E^{(0)})$ applied to the radial equation for $\tilde{V}_1$ generates

$$\tilde{\tilde{V}}_{1,4}(r)=\frac{l(l+1)}{2r^2}+U(r)-\frac{d^2}{dr^2}\ln\left(\frac{\psi^{(0)}(r)}{r^{l+1}}\right)-\frac{d^2}{dr^2}\ln[r^{l+1}\tilde{f}_1(r,E^{(0)})] \tag{53}$$

where $\tilde{f}_1(r,E^{(0)})$ is the Jost solution in the potential $\tilde{V}_1$ at energy $E^{(0)}$. $\tilde{\tilde{V}}_{1,4}$ corresponds to angular momentum l. T_4 leads to the following modification of the Jost function:

$$\frac{\tilde{\tilde{F}}_{1,4}(l,k)}{\tilde{F}_1(l+1,k)}=\frac{(k+i\gamma^{(0)})}{k}. \tag{54}$$

Equations (17) and (54) show that

$$\frac{\tilde{\tilde{F}}_{1,4}(l,k)}{F(l,k)}=\frac{(k+i\gamma^{(0)})}{(k-i\gamma^{(0)})} \tag{55}$$

which is identical in form to the expression obtained by the Gelfand–Levitan procedure, equation (52). It will now be explicitly demonstrated that the potential $\tilde{\tilde{V}}_{1,4}$ is identical to the potential obtained by the Gelfand–Levitan procedure.

The supersymmetric pairing of V and $\tilde{V}_1$ enables a simple construction of the regular solution $\tilde{\varphi}_1$ and the Jost solution $\tilde{f}_1$ for the potential $\tilde{V}_1$ at the energy $E^{(0)}<\tilde{E}_1^{(0)}$. As shown in the appendix,

$$\tilde{\varphi}_1(r,E^{(0)})\propto\frac{1}{\psi^{(0)}(r)}\int_0^r(\psi^{(0)}(z))^2\,dz \tag{56a}$$

$$\tilde{f}_1(r,E^{(0)})\propto\frac{1}{\psi^{(0)}(r)}\int_r^\infty(\psi^{(0)}(z))^2\,dz. \tag{56b}$$

Equations (53), (56b) and (1) show that

$$\tilde{\tilde{V}}_{1,4}(r)=V(r)-\frac{d^2}{dr^2}\ln\left(\int_r^\infty(\psi^{(0)}(z))^2\,dz\right). \tag{57}$$

After the application of the two transformations T_1 and T_4 the resulting spectrum is

$$\tilde{\tilde{E}}_{1,4}^{(m)}=E^{(m+1)}\qquad m=0,1,2,\ldots \tag{58a}$$

with the eigenstates

$$\tilde{\tilde{\psi}}_{1,4}^{(m)}=-(E^{(m+1)}-E^{(0)})^{-1}\tilde{A}_4^-A_1^-\psi^{(m+1)} \tag{58b}$$

where

$$\tilde{A}_4^-=\frac{1}{\sqrt{2}}\left\{-\frac{d}{dr}+\left[\frac{d}{dr}\ln\left(\frac{1}{\psi^{(0)}(r)}\int_r^\infty(\psi^{(0)}(z))^2\,dz\right)\right]\right\} \tag{58c}$$

and

$$A_1^-=\frac{1}{\sqrt{2}}\left[-\frac{d}{dr}+\left(\frac{d}{dr}\ln\psi^{(0)}(r)\right)\right]. \tag{10c}$$

Equation (58*b*) can be simplified to the form

$$\tilde{\tilde{\psi}}_{1,4}^{(m)}(r) = \psi^{(m+1)}(r) + \frac{1}{2(E^{(m+1)} - E^{(0)})} \frac{\psi^{(0)}(r)}{\int_r^\infty (\psi^{(0)}(z))^2 \, \mathrm{d}z} \times \left(\frac{\mathrm{d}}{\mathrm{d}r} \psi^{(0)}(r) \psi^{(m+1)}(r) - \psi^{(0)}(r) \frac{\mathrm{d}}{\mathrm{d}r} \psi^{(m+1)}(r) \right). \tag{59}$$

The phaseshift relation is

$$\tilde{\tilde{\delta}}_{1,4}(l, k) = \delta(l, k) - 2 \tan^{-1}(\gamma^{(0)}/k). \tag{60}$$

It is clear that $\lim_{r\to 0} \tilde{\tilde{\psi}}_{1,4}^{(m)} = \lim_{r \to 0} \psi^{(m+1)}$ showing that the bound-state normalisations are unaltered by the transformations. The expressions for the new eigenfunction $\tilde{\tilde{\psi}}_{1,4}$ and the new potential $\tilde{\tilde{V}}_{1,4}$ are identical to the results quoted by Abraham and Moses (1980). To establish this identity, it must be noted that the Wronskian between $\psi^{(0)}$ and $\psi^{(m+1)}$ may be written using the Schrödinger equation at the energies $E^{(0)}$ and $E^{(m+1)}$ in the form

$$\int_0^r \psi^{(0)}(x) \psi^{(m+1)}(x) \, \mathrm{d}x = \frac{1}{2(E^{(m+1)} - E^{(0)})} \left(\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}r} \psi^{(m+1)} - \psi^{(0)} \frac{\mathrm{d}}{\mathrm{d}r} \psi^{(m+1)} \right). \tag{61}$$

This completes the proof that the application of Gelfand-Levitan procedure to eliminate the ground state without changing the angular momentum is completely equivalent to a transformation T_1, which eliminates the ground state but changes the angular momentum from l to $(l+1)$, followed by a transformation T_4 which changes $(l+1)$ to l without changing the spectrum.

4.2. Addition of a bound state below the ground state of V by the Gelfand-Levitan procedure

The Gelfand-Levitan equations can be used to introduce a new bound state at energy $\tilde{E} < E^{(0)}$ without altering all the other bound-state energies, normalisation constants, spectral density for positive energies and angular momentum. Addition of a bound state at $\tilde{E} = \frac{1}{2}\tilde{\gamma}^2$ without altering the spectral density for positive energies corresponds to the modification of the Jost function given by

$$\frac{F_{\mathrm{new}}(l, k)}{F_{\mathrm{old}}(l, k)} = \frac{(k - \mathrm{i}\tilde{\gamma})}{(k + \mathrm{i}\tilde{\gamma})}. \tag{62}$$

The analysis in § 3 shows that the transformation implied by equation (62) may be achieved by two steps.

Step 1. Apply a transformation of type $T_3(E)$ with the choice of energy $E = \tilde{E}$. T_3 changes l to $(l+1)$ but leaves the spectrum unaltered. Starting from the potential defined in equation (1), T_3 generates

$$\tilde{V}_3(r) = \frac{(l+1)(l+2)}{2r^2} + U(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln\left(\frac{\varphi(r, \tilde{E})}{r^{l+1}} \right) \tag{35}$$

where $\varphi(r, \tilde{E})$ is the regular solution in V for the energy $\tilde{E} < E^{(0)}$. After application of T_3 the modified eigenstates and phase shifts are given by equations (36) and (37). The modification of the Jost function is given by

$$\frac{\tilde{F}_3(l+1, k)}{F(l, k)} = \frac{k}{(k + \mathrm{i}\tilde{\gamma})}. \tag{38}$$

Step 2. Add bound state to $\tilde{V}_3$ at $\tilde{E}$ by a transformation of type T_2 which reduces the angular momentum from $(l+1)$ to l. T_2 applied to the radial equation for $\tilde{V}_3$ generates

$$\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \theta) = \frac{l(l+1)}{2r^2} + U(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln\left(\frac{\varphi(r, \tilde{E})}{r^{l+1}}\right) - \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln[r^{l+1}\tilde{\psi}_3(r, \tilde{E}, \theta)] \tag{63}$$

where $\tilde{\psi}_3$ is a solution in the potential $\tilde{V}_3$ such that $1/\tilde{\psi}_3$ is normalisable and θ is the parameter, with the allowed range of values $0 < \theta < \frac{1}{2}\pi$, that controls the normalisability of $1/\tilde{\psi}_3$. After the application of T_2, the following equations result:

$$\tilde{\tilde{E}}^{(0)}_{3,2} = \tilde{E} \tag{64a}$$

$$\tilde{\tilde{\psi}}^{(0)}_{3,2}(r, \tilde{E}, \theta) = 1/\tilde{\psi}_3(r, \tilde{E}, \theta) \tag{64b}$$

and

$$\frac{\tilde{\tilde{F}}_{3,2}(l, k)}{\tilde{F}_3(l+1, k)} = \frac{(k - \mathrm{i}\tilde{\gamma})}{k}. \tag{64c}$$

Equations (38) and (64*c*) show that

$$\frac{\tilde{\tilde{F}}_{3,2}(l, k)}{F(l, k)} = \frac{(k - \mathrm{i}\tilde{\gamma})}{(k + \mathrm{i}\tilde{\gamma})} \tag{65}$$

which is identical in form to equation (62). It will now be shown that an appropriate choice of θ leads to expressions identical to those derived using the Gelfand–Levitan procedure.

It is easy to show from equation (35) that one of the solutions in the potential $\tilde{V}_3$ at energy $\tilde{E}$ is given by

$$\xi_1 = \frac{1}{\varphi(r, \tilde{E})}. \tag{66a}$$

The second linearly independent solution is given by

$$\xi_2 = \frac{1}{\varphi(r, \tilde{E})} \int_0^r (\varphi(z, \tilde{E}))^2 \, \mathrm{d}z. \tag{66b}$$

The solution $\tilde{\psi}_3$ is expressed in terms of the regular solution $\tilde{\varphi}_3$ and the Jost solution $\tilde{f}_3$ in the potential $\tilde{V}_3$ at energy $\tilde{E}$ as

$$\tilde{\psi}_3(r, \tilde{E}, \theta) = \tilde{\varphi}_3(r, \tilde{E}) \cos\theta + \tilde{f}_3(r, \tilde{E}) \sin\theta. \tag{67}$$

By studying the limiting behaviour of φ, ξ_1, ξ_2, $\tilde{\varphi}_3$ and $\tilde{f}_3$ it is easy to establish that

$$\tilde{\varphi}_3 \propto \xi_2 \tag{68a}$$

$$\tilde{f}_3 \propto \xi_1. \tag{68b}$$

Equations (64*b*), (67), (68) and (66) then show that

$$\tilde{\tilde{\psi}}^{(0)}_{3,2}(r, \tilde{E}, \theta) = \frac{(\sin\theta \cos\theta)^{1/2} \varphi(r, \tilde{E})}{(\sin\theta + \cos\theta \int_0^r \varphi^2(x, \tilde{E}) \, \mathrm{d}x)} \tag{69a}$$

where

$$\int_0^\infty (\tilde{\tilde{\psi}}^{(0)}_{3,2})^2 \, \mathrm{d}r = 1. \tag{69b}$$

Equation (69) shows that $\theta = \frac{1}{4}\pi$ corresponds to the choice of the normalisation constant for the ground state as 1. For $\theta = \frac{1}{4}\pi$, equations (63), (67), (68) and (66) show that

$$\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \tfrac{1}{4}\pi) = V(r) - \frac{d^2}{dr^2} \ln\left(1 + \int_0^r \varphi^2(x, \tilde{E})\, dx\right) \tag{70}$$

and

$$\tilde{\tilde{\psi}}_{3,2}^{(m+1)}(r, \tilde{E}, \tfrac{1}{4}\pi) = -(E^{(m)} - \tilde{E})^{-1} \tilde{A}_2^- A_3^- \psi^{(m)} \tag{71a}$$

where

$$\tilde{A}_2^-(\tilde{E}, \tfrac{1}{4}\pi) = \frac{1}{\sqrt{2}}\left\{-\frac{d}{dr} + \left[\frac{d}{dr} \ln\left(\frac{1 + \int_0^r \varphi^2(x, \tilde{E})\, dx}{\varphi(r, \tilde{E})}\right)\right]\right\} \tag{71b}$$

and

$$A_3^-(\tilde{E}) = \frac{1}{\sqrt{2}}\left[-\frac{d}{dr} + \left(\frac{d}{dr} \ln \varphi(r, \tilde{E})\right)\right]. \tag{36c}$$

Equation (71a) can be simplified to the form

$$\tilde{\tilde{\psi}}_{3,2}^{(m+1)}(r, \tilde{E}) = \psi^{(m)}(r) - \frac{1}{2(E^{(m)} - \tilde{E})} \frac{\varphi(r, \tilde{E})}{(1 + \int_0^r \varphi^2(x, \tilde{E})\, dx)}$$
$$\times \left(\frac{d\varphi(r, \tilde{E})}{dr} \psi^{(m)}(r) - \varphi(r, \tilde{E}) \frac{d\psi^{(m)}(r)}{dr}\right). \tag{72}$$

The phaseshift relation is

$$\tilde{\tilde{\delta}}_{3,2}(l, k) = \delta(l, k) + 2 \tan^{-1}(\tilde{\gamma}/k). \tag{73}$$

It is easy to show that $\lim_{r\to 0} \tilde{\tilde{\psi}}_{3,2}^{(m+1)} = \lim_{r\to 0} \psi^{(m)}$. Thus the normalisation of the eigenstates of V are left unaltered. The expressions for the new eigenfunctions $\tilde{\tilde{\psi}}_{3,2}^{(m+1)}$ and the new potential $\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \frac{1}{4}\pi)$ are identical to the results cited in Abraham and Moses (1980).

The family of potentials $\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \theta)$ in equation (63) lead to identical phaseshifts as the θ independence of equation (65) clearly shows. It is easy to show that the normalisation constants for the excited states of the potentials $\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \theta)$ for various values of the parameter θ in the range $0 < \theta < \frac{1}{2}\pi$, are the same. Equation (69) shows that the parameter θ affects the normalisation of the ground state. The family of potentials $\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \theta)$ therefore have identical spectra, identical phaseshifts and identical normalisation constants for the excited states but have different normalisation constants for the ground state. Hence the family $\tilde{\tilde{V}}_{3,2}(r, \tilde{E}, \theta)$, $0 < \theta < \frac{1}{2}\pi$ is a 'phase equivalent family'. It would also be possible to obtain an expression for the eigenfunctions for a particular value of θ directly in terms of the eigenfunctions for another value of θ. This question will be studied in the next subsection.

In this subsection it has been shown that the Gelfand–Levitan procedure for the addition of a bound state without changing the angular momentum is completely equivalent to a transformation T_3 which does not alter the spectrum but changes l to $(l+1)$ followed by a transformation T_2 which adds the new bound state and changes $(l+1)$ to l.

4.3. *The phase equivalent family*

The procedure for generating the phase equivalent family corresponds to leaving the Jost function unaltered but changing the normalisation of one or more states. The change of normalisation of the ground state can be accomplished in two steps.

Step 1. A transformation of type T_1 can be used to eliminate the ground state and change the angular momentum from l to $(l+1)$. Starting from the potential in equation (1), T_1 generates

$$\tilde{V}_1(r)=\frac{(l+1)(l+2)}{2r^2}+U(r)-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\psi^{(0)}(r)}{r^{l+1}}\right) \tag{9}$$

and leads to the modification of the eigenstates and phase shifts given by equations (10) and (12). The modification of the Jost function is given by

$$\frac{\tilde{F}_1(l+1, l)}{F(l, k)}=\frac{k}{(k-\mathrm{i}\gamma^{(0)})}. \tag{17}$$

Step 2. A transformation of type T_2 that adds a bound state at $\tilde{E}=E^{(0)}$ and changes $(l+1)$ to l can now be applied to the radial equation for $\tilde{V}_1$ to generate

$$\tilde{\tilde{V}}_{1,2}(r, E^{(0)}, \theta)=\frac{l(l+1)}{2r^2}+U(r)-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(\frac{\psi^{(0)}(r)}{r^{l+1}}\right)-\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln(r^{l+1}\tilde{\psi}_1(r, E^{(0)}, \theta)) \tag{74}$$

where $\tilde{\psi}_1(r, E^{(0)}, \theta)$ is a solution in the potential $\tilde{V}_1$ at energy $E^{(0)}<\tilde{E}_1^{(0)}$ that leads to normalisable $1/\tilde{\psi}_1$. T_2 leads to the following equations:

$$\tilde{\tilde{E}}^{(0)}_{1,2}=E^{(0)}, \tag{75a}$$

$$\tilde{\tilde{\psi}}^{(0)}_{1,2}=\frac{1}{\tilde{\psi}_1(r, E^{(0)}, \theta)} \tag{75b}$$

$$\frac{\tilde{\tilde{F}}_{1,2}(l, k)}{\tilde{F}_1(l+1, k)}=\frac{(k-\mathrm{i}\gamma^{(0)})}{k}. \tag{75c}$$

Equations (17) and (75*c*) show that

$$\frac{\tilde{\tilde{F}}_{1,2}(l, k)}{F(l, k)}=1. \tag{76}$$

The solution $\tilde{\psi}_1$ is given by

$$\tilde{\psi}_1(r, E^{(0)}, \theta)=\tilde{\varphi}_1(r, E^{(0)})\cos\theta+\tilde{f}_1(r, E^{(0)})\sin\theta \tag{77}$$

where $\tilde{\varphi}_1$ is the regular solution in $\tilde{V}_1$ at energy $E^{(0)}$ and $\tilde{f}_1$ is the Jost solution in $\tilde{V}_1$ at the same energy. Using equations (A6) and (A9) from the appendix, it is easy to see that

$$\tilde{\varphi}_1(r, E^{(0)})=\frac{1}{\psi^{(0)}(r)}\int_0^r(\psi^{(0)}(x))^2\,\mathrm{d}x \tag{78a}$$

$$\tilde{f}_1(r, E^{(0)})=\frac{1}{\psi^{(0)}(r)}\int_r^\infty(\psi^{(0)}(x))^2\,\mathrm{d}x. \tag{78b}$$

Equation (77) may then be written in terms of the parameter λ defined by

$$\tan\theta=1/(\lambda+1) \tag{79}$$

by using equation (78) in the form

$$\tilde{\psi}_1(r, E^{(0)}, \lambda) = \frac{1+\lambda \int_0^r (\psi^{(0)}(x))^2\,\mathrm{d}x}{\psi^{(0)}(r)} \qquad \infty > \lambda > -1. \tag{80}$$

Using equations (74) and (80) it is easy to see that the potential $\tilde{\tilde{V}}_{1,2}$ is given by

$$\tilde{\tilde{V}}_{1,2}(r) = V(r) - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left(1+\lambda \int_0^r (\psi^{(0)}(x))^2\,\mathrm{d}x\right) \tag{81}$$

with the normalised ground state, obtained from equations (75*b*) and (80),

$$\tilde{\tilde{\psi}}{}^{(0)}_{1,2}(r, \lambda) = \frac{(1+\lambda)^{1/2}\psi^{(0)}(r)}{[1+\lambda \int_0^r (\psi^{(0)}(x))^2\,\mathrm{d}x]}. \tag{82}$$

The excited states after the application of the two transformations are given by

$$\tilde{\tilde{\psi}}{}^{(m+1)}_{1,2}(r, \lambda) = -(E^{(m+1)} - E^{(0)})^{-1}\tilde{A}_2^- A_1^- \psi^{(m)} \qquad m = 0, 1, 2, \ldots \tag{83a}$$

where

$$\tilde{A}_2^-(E^{(0)}, \lambda) = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{1+\lambda \int_0^r (\psi^{(0)}(x))^2\,\mathrm{d}x}{\psi^{(0)}(r)}\right)\right] \tag{83b}$$

$$A_1^- = \frac{1}{\sqrt{2}}\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln \psi^{(0)}(r)\right)\right] \tag{10c}$$

which can be simplified to the form

$$\tilde{\tilde{\psi}}{}^{(m+1)}_{1,2}(r, \lambda) = \psi^{(m+1)}(r) - \frac{1}{2(E^{(m+1)} - E^{(0)})}\,\frac{\lambda\psi^{(0)}(r)}{[1+\lambda \int_0^r (\psi^{(0)}(x))^2\,\mathrm{d}x]}$$
$$\times\left(\frac{\mathrm{d}\psi^{(0)}(r)}{\mathrm{d}r}\psi^{(m+1)}(r) - \psi^{(0)}(r)\frac{\mathrm{d}}{\mathrm{d}r}\psi^{(m+1)}(r)\right). \tag{84}$$

The phaseshift relation is

$$\tilde{\tilde{\delta}}_{1,2}(l, k) = \delta(l, k). \tag{85}$$

It is easy to see that

$$\lim_{r\to 0} \tilde{\tilde{\psi}}{}^{(m+1)}_{1,2}(r, \lambda) = \lim_{r\to 0} \psi^{(m+1)}(r) \tag{86a}$$

independent of λ and

$$\lim_{r\to 0} \tilde{\tilde{\psi}}{}^{(0)}_{1,2}(r, \lambda) = (1+\lambda)^{1/2} \lim_{r\to 0} \psi^{(0)}(r). \tag{86b}$$

These results show that the family of potentials $\tilde{\tilde{V}}_{1,2}(r, \lambda)$ in equation (81) for $\infty > \lambda > -1$ have identical spectra, identical phaseshifts and identical normalisation constants for the excited states but have different normalisation constants for the ground state. Hence the family of potentials $\tilde{\tilde{V}}_{1,2}(r, \lambda)$, $\infty > \lambda > -1$ belong to a phase equivalent family. The expressions for $\tilde{\tilde{V}}_{1,2}(r, \lambda)$ and $\tilde{\tilde{\psi}}{}^{(m)}_{1,2}(r, \lambda)$ are in agreement with the results obtained by the Gelfand-Levitan procedure for changing the normalisation constant of the ground state (Abraham and Moses 1980).

We have shown that the Gelfand-Levitan procedure for changing the normalisation of the ground state without changing the angular momentum is equivalent to a transformation of type T_1 followed by a suitable transformation of type T_2.

5. Conclusions

In this paper it has been shown that the algebra of supersymmetry can be used to define four different types of transformations of the Schrödinger equation. These transformations taken together may be viewed as the building blocks by which the Bargmann class of potentials may be constructed. The four types of transformations have enough flexibility to allow the modification of a Jost function by any rational function of k. The simple applications of the Gelfand-Levitan equations such as the addition of a new ground state or the elimination of the ground state or the changing of the normalisation of the ground state may all be viewed as being made up of a suitable combination of the four types of transformations. The procedure for solving the Gelfand-Levitan equations for the above mentioned examples is straightforward. Nevertheless it is instructive to show that the same results could be derived from different points of view. In the applications discussed in the text, the representation of the new eigenfunctions in terms of a sequence of linear operators acting on the old eigenfunctions is a new representation. The question of the relation between the inverse scattering theory and supersymmetry has been raised recently (Nieto 1984). The analysis given in this paper clarifies the relationship between the two approaches.

In these two papers we have shown that the simplest representation of the supersymmetric algebra enables a fuller understanding of all one-dimensional quantum systems and provides a simple picture of certain aspects of the inverse scattering theory. Study of other representations of the algebra of supersymmetry would be fruitful.

Appendix

To find the regular and the Jost solutions φ_1 and f_1 in the potential V_1 at an energy E which is less than the ground-state energy $E_1^{(0)}$, we can imagine that V_1 is constructed by the elimination of the ground state $[E, \psi]$ of a potential V by using the supersymmetric procedure for the elimination of the ground state, i.e.

$$V_1 = V - \frac{d^2}{dr^2} \ln \psi \tag{A1}$$

where ψ is nodeless for $r > 0$ and satisfies

$$\frac{d^2\psi}{dr^2} = 2(V - E)\psi. \tag{A2}$$

A solution of

$$\frac{d^2\psi_1}{dr^2} = 2(V_1 - E)\psi_1 \tag{A3}$$

with V given by (A1) is

$$\psi_1 = 1/\psi \tag{A4}$$

as can easily be verified by direct calculation. A second linearly independent solution of (A3) is then given by

$$\psi_1' = \psi_1 \int^r dz/\psi_1^2 = (1/\psi) \int^r \psi^2\, dz. \tag{A5}$$

By construction $\int^r \psi^2\,dz$ is positive definite. From ψ_1 and ψ_1' we can construct the linear combinations

$$\begin{aligned} \xi_1 &= \frac{1}{\psi}\int_0^r \psi^2(z)\,dz \\ \xi_2 &= \frac{1}{\psi}\int_\infty^r \psi^2(z)\,dz \end{aligned} \tag{A6}$$

which are nodeless for $0 < r < \infty$. If

$$\lim_{r\to 0} \psi \sim r^l \qquad \lim_{r\to\infty} \psi \sim \exp(-\gamma r) \qquad \gamma = (-2E)^{1/2} \tag{A7}$$

then it is easy to show that

$$\begin{aligned} &\lim_{r\to 0} \xi_1 \sim r^{l+1} && \lim_{r\to\infty} \xi_1 \sim \exp(\gamma r) \\ &\lim_{r\to 0} \xi_2 \sim r^{-l} && \lim_{r\to\infty} \xi_2 \sim \exp(-\gamma r). \end{aligned} \tag{A8}$$

Comparison with the limiting behaviour of the regular and Jost solutions (see equations (18) and (20) in the main text) shows that

$$\varphi_1 \propto \xi_1 \qquad f_1 \propto \xi_2. \tag{A9}$$

The above construction of φ_1 and f_1 demonstrates that φ_1 and f_1 are positive definite for $0 < r < \infty$.

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